THE MANUSCRIPTS OF LEIBNIZ ON HIS DISCOVERY OF THE DIFFERENTIAL CALCULUS.

PART II (CONTINUED).

§§ XI—XV.

Between the date of the manuscript last considered and the one which follows there is a gap of seven months, for which Gerhardt does not appear to have found anything. This is very unfortunate; for in this interval Leibniz has attained to the important conclusion that *the true general method of tangents is by means of differences.* We saw that in November 1675 he had *started* to investigate more thoroughly the direct method of tangents; but the method is that of the auxiliary curve, and there is no indication whatever of the characteristic triangle. Does this interval correspond with the time taken by Leibniz for his final reading of Barrow from Lect. VI to Lect. X, comparing all the geometrical theorems with his own notation? Or is it only a strange coincidence that Leibniz's order is the same as that of Barrow, first the auxiliary curve, and lastly the method of differences? One could form a more definite opinion, if Leibniz had given a diagram for the first problem he considers, the one in the next following manuscript, which amounts to the differentiation of an inverse sine. Such a diagram he must have had beside him as he wrote; for I think the reader will find that he wants one to follow the argument; with the idea

of verifying this argument, I have not endeavored to supply the omission.

The consideration of the direct method of tangents is apparently, however, only as a means and not as an end; for Leibniz harks back to the inverse method, and to the catalogue of quadrible curves, which he seems to say he has in hand. It is not until November 1676 that he seems to be coming into his own; and it is not until July 1677 that he has a really definite statement of his rules. On the other hand, in July 1676, he is consistently using the differential factor with all his integrals, and before the end of that year he has the differential of a product, whether obtained as the inverse of his theorem $\int y\,dx = xy - \int x\,dy$, or by the use of the substitution $x + dx$, $y + dy$, is not certain; but this substitution appears in the manuscript for November 1676. Finally, in July 1677, appears the general idea of the substitution of other letters, in order to eliminate the difficulty caused by the appearance of the variable under a root sign or in the denominator of a fraction; and with this the whole thing is now fairly complete for all *algebraical functions*. There is as yet no equally clear method for the treatment of exponentials, logarithms, or trigonometrical functions; for the latter he refers to a geometrical diagram, strongly reminiscent of Barrow.

§ XI.

26 June, 1676.

Nova methodus Tangentium.
(New Method of Tangents.)

I have many beautiful theorems *with regard to the method of tangents both direct as well as inverse.* Descartes's method of tangents depends on finding two equal roots, and it cannot be employed, except in the case when all the undetermined quantities occurring in the work are expressible in terms of one, for instance, in terms of the abscissa.

But the true general method of tangents is by means of dif-

ferences. That is to say, the difference of the ordinates, whether direct or converging, is required. It follows that quantities that are not amenable to any other kind of calculus are amenable to the calculus of tangents, so long as their differences are known. Thus if we are given an equation in three unknowns, in which x is an abscissa, y an ordinate, and z the arc of a circle of which x is the sine of the complement, e. g., the equation $b^2y=cx^2+fz^2$. To find the next consecutive y, in place of x take $x+\beta$, and in place of z take $z-dz$, or, since $\overline{dz}=\dfrac{\beta r}{\sqrt{r^2-x^2}}$, we may take $z-\dfrac{\beta r}{\sqrt{r^2-x^2}}$;[51] hence we have

$$b^2\,(y)=cx^2+2cx\beta+c\beta^2+fz^2-\frac{2fz\beta r}{\sqrt{r^2-x^2}}+\frac{\beta^2r^2}{r^2-x^2}$$

Hence the difference between y and (y) is given by

$$\pm b^2y\mp b^2(y)=+2cx\beta-\frac{2fz\beta r}{\sqrt{r^2-x^2}}=b^2\,\overline{dy}\,;$$

Therefore $$\frac{dy}{\beta}=\frac{\mp 2cx\sqrt{r^2-x^2}\mp 2fzr}{b^2\sqrt{r^2-x^2}}=\frac{t}{y}=\frac{tb^2}{cx^2+fz^2}.$$

From this the flexure or sinuosity of the curve can be found, according as now $2cz\sqrt{r^2-x^2}$, now $2fzr$ predominates; for when they are equal, the ordinate on that side on which it was previously the greater then becomes the less. It is just the same, if several other undetermined quantities, such as logarithms and other things occur, no matter how they are affected, as for instance in the equation $b^2y=cx^2+fz^2+xzl$, where z is supposed to be an arc, and l a logarithm, x the sine of the complement of the arc, and y the number of the logarithm, b being the radius and unity, equal to r. Also it is just the same, whenever an undetermined transcendental has been derived from some dimension or quadrature that has not been investigated.[52]

For the rest, many noteworthy and useful theorems now arise from the foregoing by the inverse method of tangents. Thus general equations, or equations of any indefinite degree may be formed, at first indeed in two unknowns, x and y, only. But if in this way the matter does not work out satisfactorily, it will easily do so when

[51] In this and the following line I have corrected two obvious misprints; they are evidently not the fault of Leibniz, for the lines that follow from them are correct.

[52] There is some doubt here as to whether Leibniz could have given an example; but it must be remembered that these are practically only notes, mostly for future consideration.

the tables which I am investigating are finished; then it will be possible to take one or more other letters, and to take the difference as an arbitrary known formula, and when this is done it is certain that finally in any case a formula will be found such as is required, and in this way also a curve which will satisfy the conditions given; but in truth the description of the curve will need diagrams for these symbols, representing the sums of the arbitrarily chosen differences.

Now once a curve is found having the tangent property that we want, it will be more easy afterwards to find simpler constructions for it. We have this also as a convenient means enabling us to use many quantities that are transcendent, yet depending the one on the other, such for example as are all those that depend on the quadrature of the circle or the hyperbola. From these investigations it will also appear whether or no other quadratures can be reduced to the quadrature of the circle or the hyperbola. Lastly, since the finding of maxima and minima is useful for the inscription and circumscription of polygons, hence also, by employing these transcendent magnitudes, convergent series can be found, and in the same way their terminations; or of any quantities formed in the same way. However in that case it may not be so easy to argue about impossibility; at least indeed by the same method. Only I do not see how we can find whether from the quadrature of the circle, say, any sum can be found, when no quantity depending on the dimensions of the circle enters into the calculation.

§ XII.

July, 1676.

Methodus tangentium inversa.
[Inverse method of tangents.]

In the third volume of the correspondence of Descartes, I see that he believed that Fermat's method of Maxima and Minima is not universal; for he thinks (page 362, letter 63) that it will not serve to find the tangent to a curve, of which the property is that the lines drawn from any point on it to four given points are together equal to a given straight line.

[Thus far in Latin; Leibniz then proceeds in French.]

Mons. des Cartes (letter 73, part 3, p. 409) to Mons. de Beaune.

"I do not believe that it is in general possible to find the converse to my rule of tangents, nor of that which Mons. Fermat uses,

although in many cases the application of his is more easy than mine; but one may deduce from it *a posteriori* theorems that apply to all curved lines that are expressed by an equation, in which one of the quantities, x or y, has no more than two dimensions, even if the other had a thousand. There is indeed another method that is more general and *a priori,* namely, by the intersection of two tangents, which should always intersect between the two points at which they touch the curve, as near one another as you can imagine; for in considering what the curve ought to be, in order that this intersection may occur between the two points, and not on this side or on that, the construction for it may be found. But there are so many different ways, and I have practised them so little, that I should not know how to give a fair account of them."

Mons. des Cartes speaks with a little too much presumption about posterity; he says (page 449, letter 77) that his rule for resolving in general all problems on solids has been without comparison the most difficult to find of all things which have been discovered in geometry up to the present, and one which will possibly remain so after centuries, "unless I take upon myself the trouble of finding others" (as if several centuries would not be capable of producing a man able to do something that would be of greater moment).

(Page 459.) The question of the four spheres is one that is easy to investigate for a man who knows the calculus. It is due to Descartes, but as it is given in the book, it appears to be very prolix.

The problem on the inverse method of tangents, which Mons. des Cartes says he has solved (Vol. 3, letter 79, p. 460)

[Leibniz then continues in Latin.]

EAD is an angle of 45 degrees. ABO is a curve, BL a tangent to it; and BC, the ordinate, is to CL as N is to BJ. Then

$$\mathrm{CL} = \frac{\mathrm{BC} = ny}{\mathrm{BJ} = y - x}, \quad \mathrm{CL} = t,$$

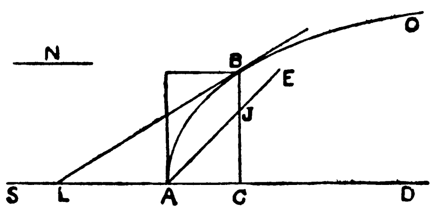
hence, $$t = \frac{ny}{y-x}, \quad \frac{n}{t} = \frac{y-x}{y} = 1 - \frac{x}{y},$$

hence, $$\frac{x}{y} = \frac{t-n}{t}; \text{ but } \frac{t}{y} = \frac{\overline{dx}}{dy};$$

therefore $$\frac{\overline{dx}}{dy} = \frac{n}{y-x}, \text{ or } \overline{dx}\,y - x\,\overline{dx} = \overline{dy}\,n;$$

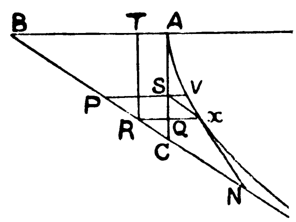
hence $$\int \overline{dx}\,y - \int x\,\overline{dx} = -n\int dy.$$

Now, $\int \overline{dy} = y$, and $\int \overline{x\,dx} = x^2/2$, and $\int \overline{dx\,y}$ is equal to the area ACBA, and the curve is sought in which the area ACBA is equal to $(x^2/2) + ny = (AC^2/2) + nBC$. [53]



Let this $x^2/2$, i. e., the triangle ACJ be cut off from the area, then the remainder AJBA should be equal to the rectangle ny.

The line that de Beaune proposed to Descartes for investigation reduces to this, that if BC is an asymptote to the curve, BA the axis, A the vertex, AB, BC, fixed lines, for BAC is at right angles.



Let RX be an ordinate, XN a tangent, then RN is always to be constant and equal to BC; required the nature of the curve.

This is how I think it should be done.

Let PV be another ordinate, differing from the other one RX by a straight line VS, found by drawing XS parallel to RN; then

[53] Leibniz has a footnote to this manuscript: "I solved in one day two problems on the inverse methods of tangents, one of which Descartes alone solved, and the other even he owned that he was unable to do."

This problem is one of them, the first mentioned in the footnote given by Leibniz. But it requires a stretch of imagination to consider Leibniz's result as a solution. For he ends up with a geometrical construction, that is at least as hard as the construction that can be made by the use of the original data. There is of course the usual misprint that one is becoming accustomed to; but there is also the unusual, for Leibniz, mistake of using his data incorrectly. Starting with the hypothesis that BC : CL = N : BJ, he writes CL = N.BC/BJ (correcting the omission of the factor N), instead of CL = BC.BJ/N.

The solution of the problem is $y + n \log(y - x + n) = 0$, as originally stated, or $x = n \log(n - y + x)$, if we continue from Leibniz's erroneous result $dx/dy = n/(y - x)$.

The point to be noted, however, is that Leibniz does not remark that "this curve appertains to a logarithm."

the triangles SVX, RXN are similar, RN $=t=c$, a constant, RX$=y$, SY$=dy$, and therefore

$$\frac{\overline{dy}}{dx} = \frac{y}{t=c}\ ;\ \text{hence}\ cy = \int \overline{y\ \overline{dx}}\quad \text{or}\ c\,\overline{dy} = y\,\overline{dx}.$$ [54]

If AQ or TR$=z$, and AC$=f$, while BC$=a$;

then, $$\frac{\text{AC}}{\text{BC}} = \frac{f}{a} = \frac{\text{TR}}{\text{BR}} = \frac{z}{x};\ \text{and thus}\ x = \frac{az}{f}.$$

If $\overline{dx}$ is constant, then $\overline{dz}$ is also constant. Hence

$c\ dy = \frac{a}{f}y\ \overline{dz}$, or $cy = \frac{a}{f}\int \overline{y\,\overline{dz}}$, and $cy\ \overline{dy} = \frac{a}{f}y^2\ \overline{dz}$, therefore

$c\frac{y^2}{2} = \frac{a}{f}\int y^2\,\overline{dz}$. Hence we have bòth the area of the figure and the moment tó a certain extent (for something must be added on account of the obliquity) ; also

$cz\ \overline{dy} = \frac{a}{f}yz\,\overline{dz}$, and therefore $c\int z\ \overline{dy} = \frac{a}{f}\int \overline{yz\ \overline{dz}}$.

Also $\frac{c\overline{dy}}{y} = \frac{a}{f}dz$, and hence, $c\int \frac{\overline{dy}}{y} = \frac{a}{f}z$. Now, unless I am greatly mistaken, $\int \frac{\overline{dy}}{y}$ is in our power.[55] The whole matter reduces to this, we must find the curve[56] in which the ordinate is such that

[54] Leibniz does not see that this result immediately gives him the equation that he requires. Thus $x = c\,\text{Log}\,y$, as he would have written it; the usual omission of the arbitrary constant does not matter in this case, so long as BA is taken as unity, which is possible with Leibniz's data.

[55] Here he seems to recognize that he has the solution. The next sentence is, however, very strange. As long ago as Nov. 1675 he has written $\int a^2/y$ as Log y, and recognized the connection between the integral and the quadrature of the hyperbola; and yet he says "unless I am mistaken, $\int dy/y$ is always in our power." Now notice that in the date there is no day of the month given, contrary to the usual custom with these manuscripts so far; can it be possible that this date was afterward added from memory, and that the manuscript should bear an earlier date? If not we must conclude that Leibniz has not yet attained to a correct idea of the meaning of his integral sign, and is still worried by the necessity (as it appears to him) of taking the y's in arithmetical progression.

[56] The passage in the original Latin is very ambiguous, and it may be that it is not quite correctly given; I think, however, that I have given the correct idea of what Leibniz intended. One has to draw an *auxiliary* curve, in which $y = dy/dx$, and then find its area; in that case it should be "divided by the differences of the abscissae" instead of "divided by the abscissae."

it is equal to the differences of the ordinates divided by the abscissae, and then find the quadrature of that figure.

$$\overline{d\sqrt{ay}} = \frac{1}{\sqrt{ay}}$$ [57]

Figures of this kind, in which the ordinates are dy/y, dy/y^2 dy/y^3, are to be sought in the same way as I have obtained those whose ordinates are $y\,dy$, $y^2\overline{dy}$, etc. Now $w/a = \overline{dy}/y$, and since $\overline{dy}$ may be taken to be constant and equal to β,[58] therefore the curve, in which $w/a = \overline{dy}/y$, will give $wy = a\beta$, which would be a hyperbola.[59] Hence the figure, in which $dy/y = z$, is a hyperbola, no matter how you express y, and if y is expressed by ϕ^2 we have $dy = 2\phi$, and $\frac{2\phi}{\phi^2} = \frac{2}{\phi}$. Now, $c\int\frac{dy}{y} = \frac{a}{f}\,z$, and therefore $\frac{fc}{a}\int\frac{1}{y} = z$, which thus appertains to a logarithm.[60]

Thus we have solved all the problems on the inverse method of tangents,[61] which occur in Vol. 3 of the Correspondence of Descartes, of which he solved one himself, as he says on page 460, letter 79, Vol. 3; but the solution is not given; the other he tried to solve but could not, stating that it was an irregular line, which in any case was not in human power, nay not within the power of the angels unless the art of describing it is determined by some other means.

§ XIII.

This manuscript bears no date: however, it was probably written very shortly after his call on Hudde at Amsterdam, on his way home from England (the second visit)

[57] An interpolated note, marking a sudden thought or guess; for the next sentence carries on the train of thought that has gone before. Query, some interval of time, either short (such as for a meal) or long (continued the next day), may have occurred here.

[58] This cannot be referred back to the present problem, since Leibniz has already assumed in it that dz and dx are constant. This may account for the fact that he has hesitated to say that the integral represents a logarithm.

[59] This working is intended to apply to the auxiliary curve mentioned above, w standing for dx, and β for dy; hence the curve is not a hyperbola; Leibniz seems to have been misled by the appearance of the equation suggesting $xy =$ constant.

[60] Here apparently he leaves the muddle, in which he has entangled himself, and returns to his original equation; he then remembers that he has found before that the integral in question leads to a logarithm.

[61] He has not solved either of them; nor can it be said from this that "Leibniz in 1676 sought and found the curve whose subtangent is constant." Of all the work that Leibniz has done hitherto, there is none that is so inconclusive as this in comparison.

to Hanover. Leibniz stayed in Holland from October 1676 to December of that year; hence the date may be fairly accurately assigned.

Hudde showed me that in the year 1662 he already had the quadrature of the hyperbola, which I found was the very same as Mercator also had discovered independently, and published. He showed me a letter written to a certain van Duck, of Leyden I think, on this subject. His method of tangents is more complete than that of Sluse, in that he is able to use any arithmetical progression, as in a simple equation, whereas Sluse and others can use only one. Hence constructions can be made simple, while terms can be eliminated at will. This also can be made use of for eliminating any letter with greater facility, for numerous equations of all sort are thereby rendered fit for elimination.

$$\begin{array}{llll}
x^3 & +\ px^2 & +\ qx & = 0 \\
 & y & y.\ y & \\
 & & y^2.\ y^2 & \\
 & & y^3 & \\
\hline
3x^2 & +\ 2px^2 & +\ qx & 0 \\
 & 2yx^2 & +\ yx & \\
 & & y^2x &
\end{array}
\qquad
\begin{array}{l}
x^2 + xy + y^2 + x + y + a = 0 \\
2x\overline{dx} + x\overline{dy} + 2y\overline{dy} + \overline{dx} + \overline{dy} \quad = 0 \\
y\overline{dx} \\
\dfrac{t}{y} = \dfrac{\overline{dx}}{dy} = \dfrac{x+2y+1}{y+2x+1}
\end{array}$$

What I had observed with regard to triangular numbers for three equal roots, and pyramidal numbers for four, was already known to him, and indeed even more generally,

$$\begin{array}{cccccccccc}
 & & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 \\
 & -3 & -1 & 0 & 0 & 1 & 3 & 6 & 10 & 15 \\
-4 & -1 & 0 & 0 & 0 & 1 & 4 & 10 & 20 &
\end{array}$$

Here it must be observed that the number of zeros increases, as this is of the greatest service in separating roots.

He has also rules for multiplying equations, so that they are not only determined for equal roots, but also for roots increasing arithmetically, or geometrically, or according to any progression.

Hudde has a most elegant construction for describing two curves, one outside and the other inside a circle, which are capable of quadrature, and by means of these curves he finds the true area of a circle so nearly, that with the help of the dodecagon, in a number of six figures, there is an error of only three units, or 3/100000.

He has a method for finding the real roots of equations, having some roots real and the rest impossible, by the help of another equation having all its roots real, and as many in number as he previously had of real and impossible together.

He had an example of a beautiful method of finding sums of series by the continuous subtractions of geometrical progressions. He subtracts geometrical progressions whose sums are also geometrical progressions, and thus he can find the sums of the sums, and so he obtains the sum of the series. This method is excellent for a series whose numerators are arithmetical, and denominators geometrical, such as,

$$\frac{1}{2}\ \frac{2}{4}\ \frac{3}{8}\ \frac{4}{16}\ \cdots\cdots\cdots .$$

He has three series like those of Wallis, for interpolations for the circle. He says that there are no more by that method, I think.

Also he can very often write down the quadratures of irrationals, as also their tangents, without eliminating irrationals, or fractions, etc.

§ XIV.

November, 1676.

Calculus Tangentium differentialis.
[Differential calculus of tangents.]

$$\overline{dx}=1,\quad \overline{dx^2}=2x,\quad \overline{dx^3}=3x^2,\quad \text{etc.}$$

$$\overline{d\frac{1}{x}}=-\frac{1}{x^2},\quad \overline{d\frac{1}{x^2}}=-\frac{2}{x^2},\quad d\frac{1}{x^3}=\frac{3}{x^2},\quad \text{etc.}$$

$$\overline{d\sqrt{x}}=\frac{1}{\sqrt{x}},\quad \text{etc.}$$

From these the following general rules may be derived for the differences and sums of the simple powers:

$$\overline{dx^e}=e,x^{e-1}, \text{ and conversely } \int\overline{x^e}=\frac{x^{e+1}}{e+1}.$$

Hence, $\overline{d\frac{1}{x^2}}=\overline{dx^{-2}}$ will be $-2x^{-3}$ or $-\frac{2}{x^3}$,

and $\overline{d\sqrt{x}}$ or $dx^{1/2}$ will be $-\frac{1}{2}x^{-1/2}$ or $-\frac{1}{2}\sqrt{\frac{1}{x}}$.

Let $y=x^2$, **then** $\overline{dy}=2x\,\overline{dx}$ or $\frac{\overline{dy}}{dx}=2\,x$.

This reasoning is general, and it does not depend on what the progression for the x's may be.[62] By the same method, the general rule is established as:

$$\frac{\overline{dx^e}}{dx} = e\,x^{e-1}, \text{ and } \int \overline{x^e\,dx} = \frac{x^{e+1}}{e+1}.$$

Suppose that we have any equation whatever, say,

$$ay^2+byx+cz^2+f^2x+g^2y+h^3=0,$$

and suppose that we write $y+dy$ for y, and $x+dx$ for x, we have, by omitting those things which should be omitted, another equation

$$\left.\begin{array}{l} ay^2+byx+cx^2+f^2x+g^2y+h^3=0 \\ \hline a2\overline{dy}y+by\overline{dx}+2cx\overline{dx}+f^2\overline{dx}+g^2\overline{dy} \\ \qquad bx\overline{dy} \\ \hline a\,\overline{dy}^2+b\,\overline{dx}\,\overline{dy}+c\,\overline{dx}^2=0 \end{array}\right\} = 0 \quad (63)$$

This is the origin of the rule published by Sluse. It can be extended indefinitely: Let there be any number of letters, and any formula composed from them; for example, let there be the formula made up of three letters,

$$ay^2 \quad bx^2 \quad cz^2 \quad fyx \quad gyx \quad hxz \quad ly \quad mx \quad nz \quad p=0.$$

From this we get another equation

$$\begin{array}{cccccccc} ay^2 & bx^2 & cz^2 & fyx & \text{simi-} & ly & mx & \text{simi-}\ p \\ \hline 2a\overline{dy}y & 2b\overline{dx}x & 2c\overline{dz}z & fy\overline{dx} & \text{larly} & l\overline{dy} & m\overline{dx} & \text{larly} \\ & & & fx\overline{dy} & & & & \\ \hline a\,\overline{dy}^2 & b\overline{dx}^2 & c\overline{dz}^2 & f\overline{dx}\,\overline{dy} & \ldots\ldots & & & \end{array}$$

It is plain from this that by the same method tangent planes

[62] AT LAST! The recognition of the fact that neither dx nor dy need necessarily be constant, and the use of another letter to stand for the function that is being differentiated, mark the beginning, the true beginning, of Leibniz's development of differentiation. Later in this manuscript we find him using the third great idea, probably suggested by the second of those given above, namely, the idea of substitution, by means of which he finally attains to the differentiation of a quotient, and a root of a function.

It is very suggestive that this remarkable advance occurs after his second visit to London, while he is staying in Holland. Did some one tell then of the work of Newton, or of Barrow's method (which is geometrically an exact equivalent of substitution), pointing out those things of which he had not perceived the drift, or is it the result of his intercourse with Hudde? For the date is that of his stay at The Hague. (For the answer to this query see an article to follow, entitled "Leibniz in London."—ED.)

[63] This is Barrow all over; even to the words *omissis omittendis* instead of Barrow's *rejectis rejiciendis*. Lect. X, Ex. 1 on the differential triangle at the end of the lecture.

to surfaces may be obtained, and in every case that it does not matter whether or no the letters x, y, z have any known relation, for this can be substituted afterward.

Further, the same method will serve admirably, even though compound fractions or irrationals enter into the calculation, nor is there any need that other equations of a higher degree should be obtained for the purpose of getting rid of them; for their differences are far better found separately and then substituted; hence the ordinary method of tangents will not only proceed when the ordinates are parallel, but it can also be applied to tangents and anything else, aye, even to those things that are related to them, such as proportions of ordinates to curves, or where the angle of the ordinates changes according to some determined law. It will be worth while especially to apply the method to irrationals and compound fractions.[64]

$$d\overline{\sqrt[2]{a+bz+cz^2}}. \quad \text{Let } a+bz+cz^2=x;$$

then $$d\overline{\sqrt[2]{x}} = -\frac{1}{2\sqrt{x}}, \text{ and } \frac{dx}{dz} = b+2cz;$$

therefore $$d\overline{\sqrt[2]{a+bz+cz^2}} = -\frac{b+2cz}{2\,dz\,\sqrt{a+bz+cz^2}}$$

Taking any equation between two letters x and y for a curve, and determining the equation of the tangent, either of the two letters x or y can be eliminated, so that all that remains is the other together with $\overline{dx}$ and $\overline{dy}$; and this will be worth while doing in all cases to facilitate the calculation.

If three letters are given, say x, y and z, and the value of $\overline{dz}$ is expressed in terms of x or y (or even of both), an equation for the tangents will at length be obtained, in which again there will be left only one or other of the letters x or y together with the two, $\overline{dx}$ and $\overline{dy}$; sometimes z itself cannot be eliminated. Also this can be deduced in all cases of an assumed value of $\overline{dz}$, and in the same way more additional letters can be taken. Thus, bringing together every general calculus into one, we obtain the most general of them all. Besides, the assumption of a large number of letters may be employed to solve problems on the inverse method of tangents, with the assistance of quadratures.

[64] Here we have the idea of substitutions, which made the Leibnizian calculus so superior to anything that had gone before. Note that he still has the erroneous sign that he obtained for the differentiation of $\sqrt{x}$ at the beginning of this manuscript. Also that the dz is wrongly placed in the denominator of the result.

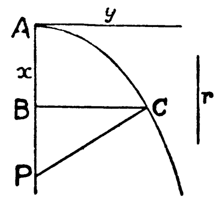
Thus, if the following problem is set for solution: It is given that the sum of the straight lines CB, BP or

$$y + y\frac{\overline{dy}}{dx} = xy\,;$$

we have

$$\overline{dx} + \overline{dy} = \overline{xdx}$$

$$\text{or } x + y = \frac{x^2}{2}.$$



Thus we have the curve in which the sum of CB + BP (multiplied by a constant r) is equal to the rectangle AB.BC.

There are two marginal notes by Leibniz that must be referred to, in this manuscript. The first reads:

It is especially to be observed about my calculus of differences that, if

$$b, ydx + xdy + \text{etc.} = 0$$

then $byx + f$ etc. $= 0$, and so on for the rest. It is to be seen what is to be done about the h^3. For the purpose of making these calculations better, the equation $ay^2 + byx + cx^2 +$ etc. can be changed into something else by means of another relation of the curve, and if it turns out all right it may be compared to another calculation of the differences, since it comes to the thing as by the first. The two points to be noticed are that Leibniz now for the first time recognizes the need of considering the arbitrary constant of integration, though he hardly grasps how it arises, and that even now he cannot refrain from harking back to his obsession of the obtaining of several equations for comparison. This note is not made any the easier to understand by its being starred by Gerhardt for reference to the differentiation of x^2, whereas it obviously (when you come later to the passage) refers to the differentiation of the equation of the second degree.

The second note refers to the substitution of $x + dx$ for x and $y + dy$ for y, and reads:

Either dx or dy can be expressed arbitrarily, a new equation being obtained; and either dx or dy being taken away, x, or y, say, can be otherwise expressed in terms of the quantities. It is not true, I think, that this is so, for then a catalogue of all curves capable of quadrature would result, by supposing one or other of them to be constant.

The point to be noticed in this rather ambiguous statement is that Leibniz is still thinking of his catalogue, and is not himself convinced of the completeness of his method for all purposes.

§ XV.

There is an interval of nearly seven months between the date of the manuscript last considered and the one that now follows. This interval has been full of work; for we now find a clear exposition of the rules for the differentia-

tion of a sum, difference, product, quotient, etc., though these are without proof, or indication of the manner in which they have been obtained. There is also no rule given for a logarithm, an exponential, or a trigonometrical ratio. Leibniz may have known them, but even then it would not be surprising to find them left out; for Leibniz's great idea was the use of his method to facilitate calculation. We must conclude therefore that these rules are a development of the method of substitution outlined in the preceding manuscript.

This essay has several peculiar characteristics of its own, which distinguish it from those that have gone before. It is written throughout in French; it is to some extent historical and critical, having the appearance of being prepared for publication, or possibly as a letter; this is corroborated by the fact that there is an original draft and a more fully detailed revision. Could it be that this is the original of Leibniz's communication of this method to Newton and others? If so, Leibniz is very careful not to give much away. The figures are strongly reminiscent of Barrow, but the context does not deal with subtangents, which are such a feature in all Barrow's work.

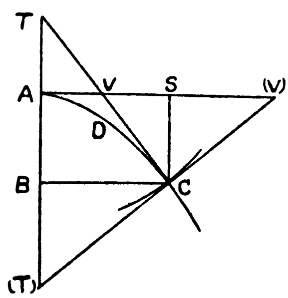
The start from the work of Sluse is peculiar; it seems to suggest that Leibniz is pointing out that his method is a fuller development of that of the former. Leibniz has already hazarded two different guesses at the origin of the rules given by Sluse; the second, namely, by substitution of $x + dx$ for x, etc., being the more probable. Is Leibniz trying to draw a red herring across the trail, the real trail that leads to Barrow's a and e?

11 July 1677.

Méthode générale pour mener les touchantes des Lignes Courbes sans calcul, et sans réduction des quantités irrationelles et rompues.

[General method for drawing tangents to curves without calculation,and without reducing irrational or fractional quantities.]

Slusius has published his method of finding tangents to curves without calculation, in which the equation is purged of irrational or fractional quantities.



For example, a curve DC being given, in which the equation expresses the relation between BC and AS, which we will call y, and AB or SC, which we will call x; let this be

$$a + bx + cy + dxy + ex^2 + fy^2 + gx^2y + hxy^2 + kx^3 + ly^3 + \text{etc.} = 0.$$

One has only to write

$$\begin{array}{l} 0 + b\xi + cv + dxv + 2ex\xi + 2fyv + gx^2v + hy^2\xi + 3kx^2\xi + 3ly^2v \\ \qquad\quad dy\xi \qquad\qquad\qquad\qquad 2gxy\xi \quad 2hxyv \end{array}$$

$$\begin{array}{l} + mx^2y^2 + nx^3y + pxy^3 + qx^4 + ry^4 \qquad (65) \\ + 2mx^2yv + nx^3v + py^3\xi + 4qx^3\xi + 4ry^3v \\ + 2mxy^2\xi + 3nx^2y\xi + 3py^2xv \end{array}$$

that is to say, if the equation is changed to a proportion,

$$\frac{\xi}{v} = \frac{c + dx + 2fy + gx^2 + 2hxy + 3ly^2 + 2mx^2y + \text{etc.}}{b + dy + 2ex + 2gxy + hy^2 + 3kx^2 + \text{etc.}};$$

and, supposing that $\frac{\xi}{v}$ expresses the ratio $\frac{\text{TB}}{\text{BC} = x}$ or $\frac{\text{CS} = y}{\text{SV}}$, then TB or SV can be obtained, if BC and SC are supposed to be given. When the given magnitudes, b, c, d, e, etc., with their proper signs, make the value of ξ/v a negative magnitude, the tangent will not be CT which goes toward A, the start of the abscissa AB, but C(T) which goes away from it. That is all that has been

[65] This line represents the "etc." of the original equation, and is set down for the purpose of getting the derived terms; the complete derived equation therefore consists of the two lines above and the two below. Note the omission of the negative sign, when changing from the equation to the proportion.

published up to the present time, easy to understand by any one that is versed in these matters. But when there are irrational or fractional magnitudes, which contain either x or y or both, this method cannot be used, except after a reduction of the given equation to another that is freed from these magnitudes. But at times this increases to a terrible degree the calculation and obliges us to rise to very high dimensions, and leads us to equations for which the process of depression is often very difficult. I have no doubt that the gentlemen[66] I have just named know the remedy that it is necessary to apply, but as it is not as yet in common use, and is I believe known to but a few, also because it gives the finishing touch to the problem that Descartes said was the most difficult to solve of all geometrical problems, because of its general utility, I have thought it a good thing to publish it.

Suppose we have any formula or magnitude or equation such as was given above,

$$a+bx+cy+dxy+ex^2+fy^2+\text{etc.};$$

for brevity let us call it ω; that which arises from it when it is treated in the manner given above, namely,

$$b\xi+cv+dxv+dy\xi+\ \text{etc.};$$

will be called $d\omega$; and in the same way, if the formula is λ or μ, then the result above will be $d\lambda$ or $d\mu$, and similarly for everything else. Now let the formula or equation or magnitude ω be equal to λ/μ, then I say that $d\omega$ will be equal to $\dfrac{\mu\overline{d\lambda}-\lambda\overline{d\mu}}{\mu^2}$. This will be sufficient to deal with fractions.

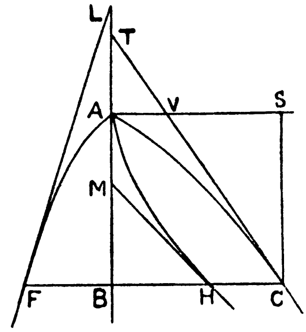
Again, let ω be equal to $\sqrt[z]{\omega}$, then $d\omega = \dfrac{dw}{z.\sqrt[z]{\omega^{z-1}}}$; and this will be sufficient for the proper treatment of irrationals.

Algorithm of the new analysis for maxima and minima, and for tangents.

Let AB $=x$, and BC $=y$, and let TVC be the tangent to the curve AC; then the ratio $\dfrac{\text{TB}}{\text{BC}=y}$ or $\dfrac{\text{SC}=x}{\text{SV}}$ will be called $\dfrac{dx}{dy}$.

[66] Leibniz, at the beginning, first wrote, "Hudde, Sluse, and others"; but later he struck out all but Sluse. (Gerhardt.)

Let there be two or more other curves, AF, AH, and suppose



that $BF = v$ and $BH = w$, and that the straight line FL is the tangent to the curve AF, and MH to the curve AH; also $\frac{LF}{FB} = \frac{dx}{dv}$, and $\frac{MH}{BH} = \frac{dx}{dw}$; then I say that dy, or dvw, will be equal to $vdw + wdv$; and if $v = w = x$, and $y = vw = x^2$, then by substituting x for v and for w, we shall have $dvw = 2xdx$.

(This will also hold good if the angle ABC is either acute or obtuse; also if it is infinitely obtuse, that is to say, if TAC is a straight line.)

[Of this rough draft there is the following revision, and this obviously comes within the same period. (Gerhardt.)]

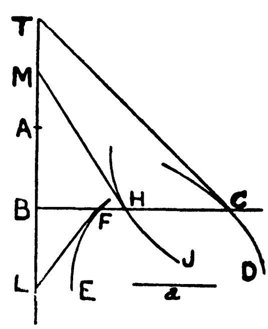
Fermat was the first to find a method which could be made general for finding the straight lines that touch analytical curves. Descartes accomplished it in another way, but the calculation that he prescribes is a little prolix. Hudde has found a remarkable abridgment by multiplying the terms of the progression by those of the arithmetical progression. He has only published it for equations in one unknown; although he has obtained it for those in two unknowns. Then the thanks of the public are due to Sluse; and after that, several have thought that this method was completely worked out. But all these methods that have been published suppose that the equation *has been reduced* and cleared of fractions and irrationals; I mean of those in which the variables occur. I however have found means of obviating these useless reductions, which make the calculation increase to a terrible degree, and oblige us to rise to very high dimensions, in which case we have to look

for a corresponding depression with much trouble; instead of all this, everything is accomplished at the first attack.

This method has more advantage over all the others that have been published, than that of Sluse has over the rest, because it is one thing to give a simple abridgment of the calculation, and quite another thing to get rid of reductions and depressions. With respect to the publication of it, on account of the great extension of the matter which Descartes himself has stated to be the most useful part of Geometry, and of which he has expressed the hope that there is more to follow—in order to explain myself shortly and clearly, I must introduce some *fresh characters,* and give to them a *new Algorithm,* that is to say, altogether special rules, for their addition, subtraction, multiplication, division, powers, roots, and also for equations.

Explanation of the characters.

Suppose that there are several curves, as CD, FE, HJ, connected with one and the same axis AB by ordinates drawn through one and the same point B to wit, BC, BF, BH. The tangents CT, FL, HM to these curves cut the axis in the points T, L, M; the



point A in the axis is fixed, and the point B changes with the ordinates. Let $AB = x$, $BC = y$, $BF = w$, $BH = v$; also let the ratio of TB to BC be called that of dx to dy, and the ratio of LB to BF that of dx to dw, and the ratio of MB to BH that of dx to dv. Then if, for example, y is equal to vw, we should say dvw instead of dy, and so on for all other cases. Let a be a constant straight line; then, if y is equal to a, that is, if CD is a straight line parallel to AB, dy or da will be equal to 0, or equal to zero. If the magnitude dx/dw comes out negative, then FL, instead of being drawn

toward A, above B, will be drawn in the contrary direction, below B.

Addition and Subtraction. Let $y = v \pm w (\pm) a$, then $\overline{dy}$ will be equal to $\overline{dv} \pm \overline{dw} (\pm) 0$.

Multiplication. Let y be equal to avw, then $\overline{dy}$ or $\overline{davw}$ or $a\,\overline{dvw}$ will be equal to $av\,\overline{dw} + aw\,\overline{dv}$.

Division. Let y be equal to $\dfrac{v}{aw}$, then $\overline{dy}$ or $\overline{d\dfrac{v}{aw}}$

$$\text{or } \frac{1}{a}\overline{d\frac{v}{w}} \text{ will be equal to } \frac{w\,\overline{dv} - v\,\overline{dw}}{aw^2}.$$

The rules for *Powers* and *Roots* are really the same thing.

Powers. If $y = w^z$, (where z is supposed to be a certain number), then $\overline{dy}$ will be equal to z, w^{z-1}, dw.

Roots or extractions. If $y = \sqrt[z]{w}$, then $\overline{dz} = \dfrac{dw}{z\sqrt[\frac{z-1}{z}]{w}}$.

Equations expressed in rational integral terms.

$$a + bv + cy + tvy + ev^2 + fy^2 + gv^2y + hvy^2 + kv^3 + ly^3$$
$$+ mv^2y^2 + nv^3y + pvy^3 + qv^4 + ry^4 = 0,$$

supposing that a, b, c, t, e, etc. are magnitudes that are known and determined; then we should have

$$\begin{aligned} 0 = b\overline{dv} + c\overline{dy} + tv\overline{dy} + 2ev\overline{dv} + 2fy\overline{dy} + gv^2\overline{dy} + hy^2\overline{dv} \\ ty\overline{dv} \qquad\qquad +2gvy\overline{dy} + 2hvy\overline{dy} \end{aligned}$$

$$\begin{aligned} + 3ly^2\overline{dy} + 2mv^2y\overline{dy} + nv^3\overline{dy} + py^3\overline{dv} + 4qv^3\overline{dv} + 4ry^3\overline{dy} \\ + 2mvy^2\overline{dv} + 3nv^2y\overline{dv} + 3py^2v\overline{dy} \end{aligned}$$

This rule can be proved and continued without limit by the preceding rules; for, if

$$a + bv + cy + tvy + ev^2 + fy^2 + gv^2y + \text{etc.} = 0,$$

then $da + dbv + dcy + tdvy + edv^2 + fdy^2 + gdv^2y +$ etc. will also be equal to 0. Now $da = 0$, $dbv = bdv$, $dcy = cdy$, $dvy = vdy + ydv$; also $dv^2 = 2vdv$, since dv^z is equal to z, v^{z-1}, dv, that is to say (by substituting 2 for z) $2vdv$; and $dv^2y = v^2dy + 2vydv$, for, supposing that $w = v^2$, then dv^2y will be dwy, and $dwy = ydw + wdy$, and dw or $dv^2 = 2vdv$; hence in the value of dwy, substituting for w and dw the values found

for them, we shall have $dv^2y = v^2dy + 2vydv$, as obtained above. This can go on without limit. If in the given equation $a + bv + cy$ + etc. $= 0$, the magnitude v were equal to x, that is to say if the line JH were a straight line which when produced passed through the point A, making an angle of 45 degrees with the axis, then the resulting equation, transformed into a proportion, would give the rule for the method of tangents, as published by Sluse; and, in consequence, this is nothing but a particular case or corollary of the general method.

Equations complicated in any manner with fractions and irrationals. These could be treated in the same way without any calculation, by supposing that the denominator of the fraction or the magnitude of which it is necessary to take the root is equal to a magnitude or letter, which is to be treated according to the preceding rules.[67]

Also, when there are magnitudes which have to be multiplied by one another, there is no need to make this multiplication in reality, which saves still more labor. One example will be sufficient.

[No example is given, however; but the following seems to have been added later, according to Gerhardt.]

Lastly this method holds good when the curves are not purely analytical, and even when their nature is not expressed by such ordinates, and in addition it gives a marvelous facility for making geometrical constructions. The true reason for an abridgment so admirable, and one that enables us to avoid reductions of fractions and irrationals, is that one can always make certain, by means of the preceding rules, that the letters dy, dv, dw, and the like, shall not occur in the denominator of the fraction, or under the root-sign.

§ XVI.

The next manuscript appears to be a more detailed revision of the one last considered. It bears no date; but it is safe to say that it belongs to a considerably later period than that of July 1677. For in this are given, by means of the *infinitely small* quantities dx and dy, proofs of the

[67] The complete statement of the method of substitutions.

fundamental rules for the first time; the figure notation is changed from the clumsy C, (C), ((C)) to the neat $_1C$, $_2C$, $_3C$; the notation for proportion is now $a:b::c:d$; and there are several other changes that readers will notice as they go along. The ideas of Leibniz are now approaching crystallization, as is evidenced by the fact that $\int y\,dx$ is clearly stated for the first time to be the sum of *rectangles made from y and dx*. It is rather astonishing, however, in this connection to find $\int \overline{x+y-v} = \int x + \int y - \int v$, which can have no significance according to the above definition; and also to find the whole thing explained by arithmetical series, in which however it is to be observed that dx is not taken to be constant. But for this one might almost place this later than the publication of the method in the *Acta Eruditorum* in 1684; in this essay Leibniz gave a full account of his rules without proofs, and is evidently trying to get away from the idea of the infinitely small, an effort which culminates in the next, and last, manuscript of this set.

If then we guess the date to be about 1680, probably we shall not be very far out.

A remarkable feature of this manuscript is the omission of really necessary figures, without which the text is very hard to follow. Of course this manuscript was written for publication, and the suggestion may be made that the diagrams were drawn separately, just as in books of that time they were printed separately on folding plates; but then, why has he given three diagrams? The only other suggestion that can be made as far as I can see is that he was referring to texts, in which the diagrams were already drawn, by Gregory St. Vincent, Cavalieri, James Gregory (one of whose theorems he quotes), Barrow (who strangely enough also quotes the very same theorem), Wallis, and others. For he mentions many of these authors, but there

is never a word about Barrow. I consider that he was looking up their theorems to show *how much superior his method was to any of theirs.*

It is to be observed that not even in this manuscript is there any mention of logarithms, exponentials, or trigonometrical ratios. We shall see later that Leibniz is reduced to obtaining the integral of $(a^2 + x^2)^{1/2}$ by reference to a figure and its quadrature; that is to say, he is apparently unable to perform the integration analytically. It therefore follows that, if he got a great deal from Barrow, he was unable to understand the Lect. XII, App. I of the *Lectiones Geometricae.*

The final conclusion that I personally have come to, after completing this examination of the manuscripts of Leibniz, as far as they are given by Gerhardt is this:

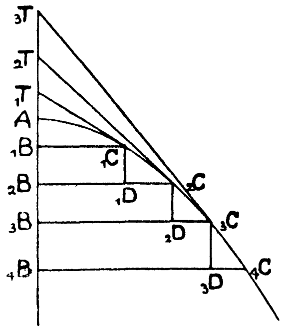
As far as the actual invention of the calculus as he understood the term is concerned, Leibniz received no help from Newton or Barrow; but for the ideas which underlay it, he obtained from Barrow a very great deal more than he acknowledged, and a very great deal less than he would like to have got, or in fact would have got if only he had been more fond of the geometry that he disliked. For, although the Leibnizian calculus was at the time of this essay far superior to that of Barrow on the question of useful application, it was far inferior in the matter of completeness.

(No date.)

Elementa calculi novi pro differentiis et summis, tangentibus et quadraturis, maximis et minimis, dimensionibus linearum, superficierum, solidorum, aliisque communem calculum transcendentibus.

[The elements of the new calculus for differences and sums, tangents and quadratures, maxima and minima, dimensions of lines, surfaces, and solids, and for other things that transcend other means of calculation.]

Let CC be a line, of which the axis is AB, and let BC be ordinates perpendicular to this axis, these being called *y*, and let AB be the abscissae cut off along the axis, these being called *x*.



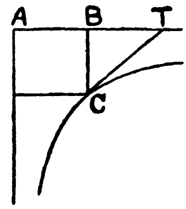
Then CD, the differences of the abscissae, will be called dx; such are $_1C_1D$, $_2C_2D$, $_3C_3D$, etc. Also the straight lines $_1D_2C$, $_2D_3C$, $_3D_4C$, the differences of the ordinates, will be called dy. If now these dx and dy are taken to be infinitely small, or the two points on the curve are understood to be at a distance apart that is less than any given length, i. e., if $_1D_2C$, $_2D_3C$, etc. are considered as the momentaneous increments[68] of the line BC, increasing continuously as it descends along AB, then it is plain that the straight line joining these two points, $_2C_1C$ say, (which is an element of the curve or a side of the infinite-angled polygon that stands for the curve), when produced to meet the axis in $_1T$, will be the tangent to the curve, and $_1T_1B$ (the interval between the ordinate and the tangent, taken along the axis) will be to the ordinate $_1B_1C$ as $_1C_1D$ is to $_1D_2C$; or, if $_1T_1B$ or $_2T_2B$, etc. are in general called t, then $t:y :: dx:dy$. Thus to find the differences of series is to find tangents.

For example, it is required to find the tangent to the hyperbola. Here, since $y=\frac{aa}{x}$, supposing that in the diagram, x stands for AB the abscissa along an asymptote, and a for the side of the power, or of the area of the rectangle AB.BC; then

$$dy = -\frac{aa}{xx}dx,$$

[68] Leibniz has evidently seen Newton's work at the time of this composition; also the use of the word "descends" in the next line again suggests Barrow, while the figure is exactly like the top half of the diagram given by Barrow for Lect. XI, 10, which is the theorem of Gregory that is quoted by Leibniz also. For this figure, see the note to that passage.

as will be soon seen when we set forth the method of this calculus; hence $dx:dy$ or $t:y :: -xx:aa :: -x : \frac{aa}{x} :: -x:y$; therefore $t=-y$,



that is, in the hyperbola BT will be equal to AB, but on account of the sign $-x$, BT must be taken not toward A but in the opposite direction.

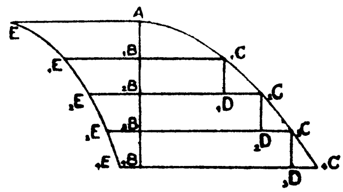
Moreover, differences are the opposite to sums; thus ${}_4B\,{}_4C$ is the sum of all the differences such as ${}_3D\,{}_4C$, ${}_2D\,{}_3C$, etc. as far as A, even if they are infinite in number. This fact I represent thus, $\int dy=y$. Also I represent the area of a figure by the sum of all the rectangles contained by the ordinates and the differences of the abscissae, i. e., by the sum ${}_1B\,{}_1D+{}_2B\,{}_2D+{}_3B\,{}_3D+$etc. For the narrow triangles ${}_1C\,{}_1D\,{}_2C$, ${}_2C\,{}_2D\,{}_3C$, etc., since they are infinitely small compared with the said rectangles, may be omitted without risk; and thus I represent in my calculus the area of the figure by $\int y\,dx$, or the sum of the rectangles contained by each y and the dx that corresponds to it; here, if the dx's are taken equal to one another, the method of Cavalieri is obtained.

But we, now mounting to greater heights, obtain the area of a figure by finding the figure of its summatrix or quadratrix; and of this indeed the ordinates are to the ordinates of the given figure in the ratio of sums to differences; for instance, let the curve of the figure required to be squared be EE, and let the ordinates to it, EB, which we will call e, be proportional to the differences of the ordinates BC, or to dy; that is let ${}_1B\,{}_1E:{}_2B\,{}_2E :: {}_1D\,{}_2C:{}_2D\,{}_3C$, and so on; or again, let $A\,{}_1B:{}_1B\,{}_1C$, ${}_1C\,{}_1D:{}_1D\,{}_2C$, etc., or $dx:dy$ be in the ratio of a constant or never-varying straight line a to ${}_1B\,{}_1E$ or e; then we have

$$dx:dy :: a:e, \text{ or } e\,dx=a\,dy;$$
$$\therefore \int e\,dx=\int a\,dy.$$

But $e\,dx$ is the same as e multiplied by its corresponding dx, such as the rectangle ${}_3B\,{}_4E$, which is formed from ${}_3B\,{}_3E$ and ${}_3B\,{}_4B$; hence, $\int e\,dx$ is the sum of all such rectangles, ${}_3B\,{}_4E+{}_2B\,{}_1E+{}_3B\,{}_2E$ + etc., and this sum is the figure $A\,{}_4B\,{}_4EA$, if it is supposed that the

dx's, or the intervals between the ordinates *e*, or BC, are infinitely small. Again, $a\,dy$ is the rectangle contained by a and dy, such as is contained by ${}_3D_4C$ and the constant length a, and the sum of



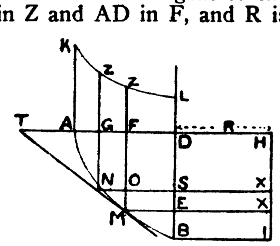
these rectangles, namely $\int a\,dy$, or ${}_3D_4C.a + {}_2D_3C.a + {}_1D_2C.a +$ etc. is the same as ${}_3D_4C + {}_2D_3C + {}_1D_2C +$ etc. into a, that is, the same as ${}_4B_4C.a$; therefore we have $\int a\,dy = a\int dy = ay$. Therefore $\int e\,dx = ay$, that is, the area A_4B_4EA will be equal to the rectangle contained by ${}_4B_4C$ and the constant line a, and generally ABEA is equal to the rectangle contained by BC and a.[69]

Thus, for quadratures it is only necessary, being given the line EE, to find the summatrix line CC, and this indeed can always be found by calculus, whether such a line is treated in ordinary geometry or whether it is transcendent and cannot be expressed by algebraical calculation; of this matter in another place.

Now the triangle for the line I call the characteristic of the line, because by its most powerful aid there can be found theorems about the line which are seen to be admirable, such as its length, the surface and solid produced by its rotation, and its center of gravity; for ${}_1C_2C$ is equal to $\sqrt{dx.dx + dy.dy}$. From this we have

[69] Leibniz does not give a diagram, but it is not difficult to construct his figure from the enunciation that he gives for it. The whole of this paragraph should be compared with the following extract from Barrow (Lect. XI, 19), piece by piece.

"Again, let AMB be a curve of which the axis is AD and let BD be perpendicular to AD; also let KZL be another line such that, when any point M is taken in the curve AB, and through it are drawn MT a tangent to the curve AB, and MFZ parallel to DB, cutting KZ in Z and AD in F, and R is a line of given length, TF : FM = R : FZ. Then the space ADLK is equal to the rectangle contained by R and DB.



For, if DH = R and the rectangle BDHI is completed, and MN is taken to be an indefinitely small arc of the curve AB, and MEX, NOS are drawn parallel to AD; then we have NO : MO = TF : FM = R : FZ;

NO.FZ = MO.R and FG.FZ = ES.EX.

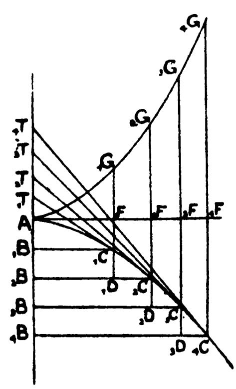
Hence, since the sum of such rectangles as FG.FZ differs only in the least degree from the space ADLK, and the rectangles ES.EX form the rectangle DHIB, the theorem is quite obvious.

at once a method for finding the length of a curve by means of some quadrature; e. g., in the case of the parabola, if $y=\frac{xx}{2a}$, then we have $dy=\frac{xdx}{a}$, and hence $_1C\,_2C=\frac{dx}{a}\sqrt{aa+xx}$; hence, $_1C\,_2C:dx$ as the ordinate of the hyperbola $\sqrt{aa+xx}$ is to the constant line a; that is, $\frac{1}{a}\int dx\sqrt{aa+xx}$, a straight line equal to the arc of a parabola, depends on the quadrature of the hyperbola, as has already been found by others; and thus we can derive by the calculus all the most beautiful results discovered by Huygens, Wallis, van Huraet, and Neil.[70]

I said above that $t:y::dx:dy$; hence we have $t\,dy=y\,dx$, and therefore $\int t\,dy=\int y\,dx$. This equation, enunciated geometrically, gives an elegant theorem due to Gregory.[71] namely that, if BAF is a right angle, and AF = BG, and FG is parallel to AB and equal to BT, that is, $_1F\,_1G={}_1B\,_1T$, then $\int t\,dy$, or the sum of the rectangles contained by t (e.g., $_4F\,_4G$ or $_4B\,_4T$) and dy ($_3F\,_4F$ or $_3D\,_4C$) is equal to the rectangles $_4F\,_3G+{}_3F\,_2G+{}_2F\,_1G+$etc., or the area of the

[70] All the things given are to be found in Barrow, but his name is not even mentioned.

[71] This is the strangest coincidence of all! For, Barrow also quotes this very same theorem of Gregory, and no other theorem; also it occurs in this very same Lect. XI that has been referred to already! *Leibniz does not give a diagram; nor from his enunciation could I complete the figure required, until I had referred to the figure given by Barrow*!!! The two diagrams are given below for comparison, Barrow's figure being the one referred to in the note above. Query, is Leibniz's figure taken from Gregory's original, which I have not been able to see, or is it the Leibnizian variation of Barrow's?



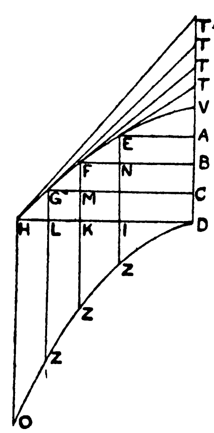


figure A_4F_4GA is equal to $\int y\,dx$, that is, to the figure A_4B_4CA; or generally, the figure AFGA is equal to the figure ABCA.

Again, other things, which are immediately evident on inspection, from a figure, are readily deduced by the calculus; for instance, in the case of the trilinear figure ABCA, the figure ABCA together with its complementary figure AFCA is equal to the rectangle ABCF, for the calculus readily shows that $\int y\,dx + \int x\,dy = xy$.

If it is required to find the volume of the solid formed by rotation round an axis, it is only necessary to find $\int y^2\,dx$; for the solid formed by a rotation round the base, $\int x^2 dy$; for the moment about the vertex, $\int yx\,dx$; and these things serve to find the center of gravity of a figure, and also give the frusta of Gregory St. Vincent, and all that Pascal, Wallis, De Laloubère, and others have found out about these matters.

For, if it is required to find the centers of lines, or the surfaces generated by their rotation, e. g., the surface generated by the rotation of the line AC about AB, it is only necessary to find

$$\overline{\int y \sqrt{dx.\,dx + dy.\,dy}}$$

or the sum of every PC applied to the axis at the point B that corresponds to it, (thus ${}_2P\,{}_2C$ will be applied perpendicular to the axis AB at ${}_2B$), producing in this way a figure of which the above represents the area. Thus the whole thing will immediately reduce to the quadrature of some plane figure, if, instead of y and dy, their values, obtained from the nature of the ordinates and the tangents to the curve, are substituted. Thus, in the case of the parabola, if y is equal to $\sqrt{2ax}$, then $dy = \dfrac{adx}{y}$ (as will be seen directly); hence we get

$$\overline{\int y \sqrt{dxdx + \frac{aa}{yy}\,dxdx}} \text{ or } \overline{\int dx\sqrt{yy + aa}} \text{ or } \overline{\int dx\sqrt{2ax + aa}}\,,$$

which depends on the quadrature of the parabola (for every $\sqrt{2ax+aa}$ or PC can be applied to a parabola, if it is supposed that AC is the parabola, and AB its axis, provided in that case the figure is changed and the curve turns its concavity toward the axis);[72] and this may be obtained by ordinary geometry, and there-

[72] The Latin here is rather ambiguous; query, a misprint. But I think I have correctly rendered the argument. It is to be noted that the parabola was at this period always thought of in the form we should now denote by the equation $y = x^2$, and the figure referred to by Leibniz is that which Wallis calls the complement of the semiparabola.

fore also a circle will be found equal to the surface of the parabolic conoid; but this is not the place to deduce it at full length.

Now these, which may seem to be great matters, are only the very simplest results to be obtained by this calculus; for many much more important consequences follow from it, nor does there occur any simple problem in geometry, either pure or applied to mechanics, that can altogether evade its power. Now we will expound the elements of the calculus itself.

The fundamental principle of the calculus.

Differences and sums are the inverses of one another, that is to say, the sum of the differences of a series is a term of the series, and the difference of the sums of a series is a term of the series; and I enunciate the former thus, $\int dx = x$, and the latter thus, $d\int x = x$.

Thus, let the differences of a series, the series itself, and the sums of the series, be, let us say,

Diffs.		1	2	3	4	5	....	dx
Series	0	1	3	6	10	15	....	x
Sums		0	1	4	10	20	25	.. $\int x$

Then the terms of the series are the sums of the differences, or $x = \int dx$; thus, $3 = 1 + 2$, $6 = 1 + 2 + 3$, etc.; on the other hand, the differences of the sums of the series are terms of the series, or $d\int x = x$; thus, 3 is the difference between 1 and 4, 6 between 4 and 10.

Also $da = 0$, if it is given that a is a constant quantity, since $a - a = 0$.

Addition and Subtraction.

The difference or sum of a series, of which the general term is made up of the general terms of other series by addition or subtraction, is made up in exactly the same manner from the differences or sums of these series; or

$$x + y - v = \int \overline{dx + dy - dv}, \quad \int \overline{x + y - v} = \int x + \int y - \int v.$$

This is evident at sight, if you take any three series, set out their sums and their differences, and take them together correspondingly as above.

Simple Multiplication.

Here $dxy = xdx + ydy$, or $xy = \int xdx + \int ydy$.

This is what we said above about figures taken together with their complements being equal to the circumscribed rectangle. It is demonstrated by the calculus as follows:

dxy is the same thing as the difference between two successive xy's; let one of these be xy, and the other $x + dx$ into $y + dy$; then we have

$$dxy = \overline{\overline{x + dx} \,.\, \overline{y + dy}} - xy = x\,dy + y\,dx + dx\,dy;$$

the omission of the quantity $dx\,dy$, which is infinitely small in comparison with the rest, for it is supposed that dx and dy are infinitely small (because the lines are understood to be continuously increasing or decreasing by very small increments throughout the series of terms), will leave $x\,dy + y\,dx$; the signs vary according as y and x increase together, or one increases as the other decreases; this point must be noted.

Simple Division.

Here we have $d\dfrac{y}{x} = \dfrac{x\,dy - y\,dx}{xx}$.

For, $d\dfrac{y}{x} = \dfrac{y + dy}{x + dx} - \dfrac{y}{x} = \dfrac{x\,dy - y\,dx}{xx + x\,dx}$, which becomes (if we write xx for $xx + x\,dx$, since $x\,dx$ can be omitted as being infinitely small in comparison with xx) equal to $\dfrac{x\,dy - y\,dx}{xx}$; also, if $y = aa$, then $dy = 0$, and the result becomes $-\dfrac{aadx}{xx}$, which is the value we used a little while before in the case of the tangent to the hyperbola.

From this any one can deduce by the calculus the rules for *Compound Multiplication and Division*; thus,

$$dxvy = xy\,dv + xv\,dy + yv\,dx,$$

$$d\frac{y}{vz} = \frac{xv\,dy - yv\,dz - yz\,dv}{vv.zz};$$

as can be proved from what has gone before; for we have

$$d\frac{y}{x} = \frac{x\,dy - y\,dx}{xx};$$

hence, putting zv for x, and $z\,dv + v\,dz$ for dx or dzv in the above, we obtain what was stated.

Powers follow: $dx^2 = 2x\,dx$, $dx^3 = 3x^2\,dx$, and so on. For, putting $y = x$, and $v = x$, we can write dx^2 for dxy, and this is (from above) equal to $x\,dy + y\,dx$, or (if $x = y$, and consequently $dx = dy$) equal to $2x\,dx$. Similarly, for dx^3 we write $dxyv$, that is (from above) $xy\,dv + xv\,dy + yv\,dx$, or (putting x for y and v and dx for dy and dv) equal to $3x^2dx$. Q. E. D. By the same method, in general, $dx^e = e.x^{\underline{e-1}}\,dx$, as can easily be proved from what has been said.

Hence also, $$d\frac{1}{x^h} = -\frac{h\,dx}{x^{\underline{h+1}}}.$$

For, if $\frac{1}{x^h} = x^e$, then $e = -h$, and $x^{e-1} = \frac{1}{x^{\underline{h+1}}}$, as is well known to any one who understands the nature of the exponents in a geometrical progression. The same thing will do for *fractions.* The procedure is the same for irrationals or *Roots.* $d\sqrt[r]{x^h} = dx^{h:r}$, (where by $h:r$ I mean h/r, or h divided by r), or dx^e (taking e equal to h/r), or $e.x^{\underline{e-1}}\,dx$, by what has been said above, or (by substituting once more $h:r$ for e, and $\overline{h-r}:r$ for $e-1$) $\frac{h}{r}.\,x^{\overline{h-r}:r}.dx$; and thus finally we get the value of $d\sqrt[r]{x^h}$.

Moreover, conversely, we have

$$\int x^e\,dx = \frac{x^{\underline{e+1}}}{e+1}, \int \frac{1}{x^e}dx = -\frac{1}{\mathrm{e}-1.x^{\underline{e-1}}}, \int \sqrt[r]{x^h}.dx = \frac{r}{r+h}\sqrt[r]{x^{\underline{\overline{h+r}:r}}}.$$

These are the elementary principles of the differential and summatory calculus, by means of which highly complicated formulas can be dealt with, not only for a fraction or an irrational quantity, or anything else; but also an indefinite quantity, such as x or y, or any other thing expressing generally the terms of any series, may enter into it.

§ XVII.

The next manuscript bears no date; but this can be easily assigned to a certain extent, from internal evidence. It is for one thing later than the publication in the *Acta Eruditorum* of Leibniz's first communication to the world of his calculus in 1684. The manuscript is an answer, or rather the first rough draft probably of such an answer, to the animadversions of Bernhard Nieuwentijt against the idea of the infinitesimal calculus. The latter stated that (i) Leibniz could explain no more than Barrow or

Newton how the infinitely small differences differed from absolute zero; (ii) it was not clear how the differentials of higher order were obtained from those of the first order; (iii) the differential method cannot be applied to exponential functions. Leibniz answers the first point skilfully, fails over the second through erroneous work, which I think he afterward perceived; for he has a note that the whole thing is to be carefully revised before publication. It almost seems that he was not quite confident in his own powers of completely answering these objections, for he also notes that the rudeness of language in which the answer is commenced must be mollified.

On the third point he is silent; in the later written *Historia,* we have seen he is able to get, not over, but round the difficulty of the exponential function; but the silence here would seem to say that Leibniz could not manage exponentials as yet.

The success of the answer to the first point is due to the underlying principle that the ratio $dy : dx$ ultimately becomes a *rate*; when this idea is muddled by an admixture of the infinitesimal idea in the last paragraph the result is almost disastrous. Leibniz, however, looked on his calculus as a tried tool more than anything else.

When my infinitesimal calculus, which includes the calculus of differences and sums, had appeared and spread, certain over-precise veterans began to make trouble; just as once long ago the Sceptics opposed the Dogmatics, as is seen from the work of Empicurus against the mathematicians (i. e., the dogmatics), and such as Francisco Sanchez, the author of the book *Quod nihil scitur,* brought against Clavius; and his opponents to Cavalieri, and Thomas Hobbes to all geometers, and just lately such objections as are made against the quadrature of the parabola by Archimedes by that renowned man, Dethlevus Cluver. When then our method of infinitesimals, which had become known by the name of the calculus of differences, began to be spread abroad by several examples of its use, both of my own and also of the famous brothers Bernoulli, and more espe-

cially by the elegant writings of that illustrious Frenchman, the Marquis d'Hopital, just lately a certain erudite mathematician, writing under an assumed name in the scientific *Journal de Trevoux,* appeared to find fault with this method. But to mention one of them by name, even before this there arose against me in Holland Bernard Nieuwentiit, one indeed really well equipped both in learning and ability, but one who wished rather to become known by revising our methods to some extent than by advancing them. Since I introduced not only the first differences, but also the second, third and other higher differences, inassignable or incomparable with these first differences, he wished to appear satisfied with the first only; not considering that the same difficulties existed in the first as in the others that followed, nor that wherever they might be overcome in the first, they also ceased to appear in the rest. Not to mention how a very learned young man, Hermann of Basel, showed that the second and higher differences were avoided by the former in name only, and not in reality; moreover, in demonstrating theorems by the legitimate use of the first differences, by adhering to which he might have accomplished some useful work on his own account, he fails to do so, being driven to fall back on assumptions that are admitted by no one; such as that something different is obtained by multiplying 2 by m and by multiplying m by 2; that the latter was impossible in any case in which the former was possible; also that the square or cube of a quantity is not a quantity or Zero.

In it, however, there is something that is worthy of all praise, in that he desires that the differential calculus should be strengthened with demonstrations, so that it may satisfy the rigorists; and this work he would have procured from me already, and more willingly, if, from the fault-finding everywhere interspersed, the wish had not appeared foreign to the manner of those who desire the truth rather than fame and a name.

It has been proposed to me several times to confirm the essentials of our calculus by demonstrations, and here I have indicated below its fundamental principles, with the intent that any one who has the leisure may complete the work. Yet I have not seen up to the present any one who would do it. For what the learned Hermann has begun in his writings, published in my defence against Nieuwentiit, is not yet complete.

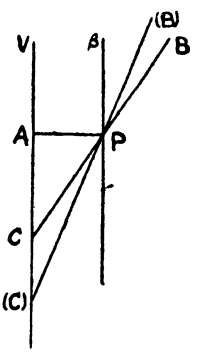
For I have, beside the mathematical infinitesimal calculus, a method also for use in Physics, of which an example was given in

the *Nouvelles de la République des Lettres*; and both of these I include under the Law of Continuity; and adhering to this, I have shown that the rules of the renowned philosophers Descartes and Malebranche were sufficient in themselves to attack all problems on Motion.

I take for granted the following postulate:

In any supposed transition, ending in any terminus, it is permissible to institute a general reasoning, in which the final terminus may also be included.

For example, if A and B are any two quantities, of which the former is the greater and the latter is the less, and while B remains the same, it is supposed that A is continually diminished, until A becomes equal to B; then it will be permissible to include under a general reasoning the prior cases in which A was greater than B, and also the ultimate case in which the difference vanishes and A is equal to B. Similarly, if two bodies are in motion at the same time, and it is assumed that while the motion of B remains the same, the velocity of A is continually diminished until it vanishes altogether, or the speed of A becomes zero; it will be permissible to include this case with the case of the motion of B under one general reasoning. We do the same thing in geometry, when two



straight lines are taken, produced in any manner, one VA being given in position or remaining in the same site, the other BP passing through a given point P, and varying in position while the point P remains fixed; at first indeed converging toward the line VA and meeting it in the point C; then, as the angle of inclination VCA is continually diminished, meeting VA in some more remote point (C), until at length from BP, through the position (B)P, it comes

to βP, in which the straight line no longer converges toward VA, but is parallel to it, and C is an impossible or imaginary point. With this supposition it is permissible to include under some one general reasoning not only all the intermediate cases such as (B)P but also the ultimate case βP.

Hence also it comes to pass that we include as one case ellipses and the parabola, just as if A is considered to be one focus of an ellipse (of which V is the given vertex), and this focus remains fixed, while the other focus is variable as we pass from ellipse to ellipse, until at length (in the case when the line BP, by its intersection with the line VA, gives the variable focus) the focus C becomes evanescent[73] or impossible, in which case the ellipse passes into a parabola. Hence it is permissible with our postulate that a parabola should be considered with ellipses under a common reasoning. Just as it is common practice to make use of this method in geometrical constructions, when they include under one general construction many different cases, noting that in a certain case the converging straight line passes into a parallel straight line, the angle between it and another straight line vanishing.

Moreover, from this postulate arise certain expressions which are generally used for the sake of convenience, but seem to contain an absurdity, although it is one that causes no hindrance, when its proper meaning is substituted. For instance, we speak of an imaginary point of intersection as if it were a real point, in the same manner as in algebra imaginary roots are considered as accepted numbers. Hence, preserving the analogy, we say that, when the straight line BP ultimately becomes parallel to the straight line VA, even then it converges toward it or makes an angle with it, only that the angle is then infinitely small; similarly, when a body ultimately comes to rest, it is still said to have a velocity, but one that is infinitely small; and when one straight line is equal to another, it is said to be unequal to it, but that the difference is infinitely small; and that a parabola is the ultimate form of an ellipse, in which the second focus is at an infinite distance from the given focus nearest to the given vertex, or in which the ratio of PA to AC, or the angle BCA, is infinitely small.

Of course it is really true that things which are absolutely equal have a difference which is absolutely nothing; and that straight lines which are parallel never meet, since the distance

[73] The term is here used with the idea of "vanishing into the far distance."

between them is everywhere the same exactly; that a parabola is not an ellipse at all, and so on. Yet, a state of transition may be imagined, or one of evanescence, in which indeed there has not yet arisen exact equality or rest or parallelism, but in which it is passing into such a state, that the difference is less than any assignable quantity; also that in this state there will still remain some difference, some velocity, some angle, but in each case one that is infinitely small; and the distance of the point of intersection, or the variable focus, from the fixed focus will be infinitely great, and the parabola may be included under the heading of an ellipse (and also in the some manner and by the same reasoning under the heading of a hyperbola), seeing that those things that are found to be true about a parabola of this kind are in no way different, for any construction, from those which can be stated by treating the parabola rigorously.

Truly it is very likely that Archimedes, and one who seems so have surpassed him, Conon, found out their wonderfully elegant theorems by the help of such ideas; these theorems they completed with *reductio ad absurdum* proofs, by which they at the same time provided rigorous demonstrations and also concealed their methods. Descartes very appropriately remarked in one of his writings that Archimedes used as it were a kind of metaphysical reasoning (Caramuel would call it metageometry), the method being scarcely used by any of the ancients (except those who dealt with quadratrices); in our time Cavalieri has revived the method of Archimedes, and afforded an opportunity for others to advance still further. Indeed Descartes himself did so, since at one time he imagined a circle to be a regular polygon with an infinite number of sides, and used the same idea in treating the cycloid; and Huygens too, in his work on the pendulum, since he was accustomed to confirm his theorems by rigorous demonstrations; yet at other times, in order to avoid too great prolixity, he made use of infinitesimals; as also quite lately did the renowned La Hire.

For the present, whether such a state of instantaneous transition from inequality to equality, from motion to rest, from convergence to parallelism, or anything of the sort, can be sustained in a rigorous or metaphysical sense, or whether infinite extensions successively greater and greater, or infinitely small ones successively less and less, are legitimate considerations, is a matter that I own to be possibly open to question; but for him who would discuss these matters, it is not necessary to fall back upon metaphysical

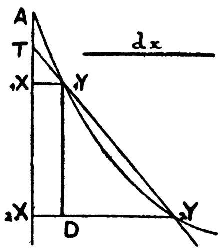
controversies, such as the composition of the continuum, or to make geometrical matters depend thereon. Of course, there is no doubt that a line may be considered to be unlimited in any manner, and that, if it is unlimited on one side only, there can be added to it something that is limited on both sides. But whether a straight line of this kind is to be considered as one whole that can be referred to computation, or whether it can be allocated among quantities which may be used in reckoning, is quite another question that need not be discussed at this point.

It will be sufficient if, when we speak of infinitely great (or more strictly unlimited), or of infinitely small quantities (i. e., the very least of those within our knowledge), it is understood that we mean quantities that are indefinitely great or indefinitely small, i. e., as great as you please, or as small as you please, so that the error that any one may assign may be less than a certain assigned quantity. Also, since in general it will appear that, when any small error is assigned, it can be shown that it should be less, it follows that the error is absolutely nothing; an almost exactly similar kind of argument is used in different places by Euclid, Theodosius and others; and this seemed to them to be a wonderful thing, although it could not be denied that it was perfectly true that, from the very thing that was assumed as an error, it could be inferred that the error was non-existent. Thus, by infinitely great and infinitely small, we understand something indefinitely great, or something indefinitely small, so that each conducts itself as a sort of class, and not merely as the last thing of a class. If any one wishes to understand these as the ultimate things, or as truly infinite, it can be done, and that too without falling back upon a controversy about the reality of extensions, or of infinite continuums in general, or of the infinitely small, ay, even though he think that such things are utterly impossible; it will be sufficient simply to make use of them as a tool that has advantages for the purpose of the calculation, just as the algebraists retain imaginary roots with great profit. For they contain a handy means of reckoning, as can manifestly be verified in every case in a rigorous manner by the method already stated.

But it seems right to show this a little more clearly, in order that it may be confirmed that the algorithm, as it is called, of our differential calculus, set forth by me in the year 1684, is quite reasonable. First of all, the sense in which the phrase "dy is the

element of y," is to be taken will best be understood by considering a line AY referred to a straight line AX as axis.

Let the curve AY be a parabola, and let the tangent at the vertex A be taken as the axis. If AX is called x, and AY, y, and the latus-rectum is a, the equation to the parabola will be $xx=ay$, and this holds good at every point. Now, let $A\,{}_1X=x$, and ${}_1X\,{}_1Y=y$



and from the point ${}_1Y$ let fall a perpendicular ${}_1YD$ to some greater ordinate ${}_2X\,{}_2Y$ that follows, and let ${}_1X\,{}_2X$, the difference between $A\,{}_1X$ and $A\,{}_2X$, be called dx; and similarly, let $D\,{}_2Y$, the difference between ${}_1X\,{}_1Y$ and ${}_2X\,{}_2Y$, be called dy.

Then, since $y=xx:a$, by the same law, we have

$$y+dy=xx+2x\,dx+dx\,dx,:a;$$

and taking away the y from the one side and the $xx:a$ from the other, we have left

$$dy:dx=2x+dx:a;$$

and this is a general rule, expressing the ratio of the difference of the ordinates to the difference of the abscissae, or, if the chord ${}_1Y\,{}_2Y$ is produced until it meets the axis in T, then the ratio of the ordinate ${}_1X\,{}_1Y$ to $T\,{}_1X$, the part of the axis intercepted between the point of intersection and the ordinate, will be as $2x+dx$ to a. Now, since by our postulate it is permissible to include under the one general reasoning the case also in which the ordinate ${}_2X\,{}_2Y$ is moved up nearer and nearer to the fixed ordinate ${}_1X\,{}_1Y$ until it ultimately coincides with it, it is evident that in this case dx becomes equal to zero and should be neglected, and thus it is clear that, since in this case $T\,{}_1Y$ is the tangent, ${}_1X\,{}_1Y$ is to $T\,{}_1X$ as $2x$ is to a.

Hence, it may be seen that there is no need in the whole of our differential calculus to say that those things are equal which have a difference that is infinitely small, but that those things can be taken as equal that have not any difference at all, provided that the calculation is supposed to be general, including both the cases in which there is a difference and in which the difference is zero;

and provided that the difference is not assumed to be zero until the calculation is purged as far as is possible by legitimate omissions, and reduced to ratios of non-evanescent quantities, and we finally come to the point where we apply our result to the ultimate case.

Similarly, if $x^3 = aay$, then we have

$$x^3 + 3xx\,dx + 3x\,dx\,dx + dx\,dx\,dx = aay + aa\,dy,$$

or cancelling from each side,

$$3xx\,dx + 3x\,dx\,dx + dx\,dx\,dx = aa\,dy,$$

or $$3xx + 3x\,dx + dx\,dx, : aa = dy : dx = {}_1X\,{}_1Y : T\,{}_1X\,;$$

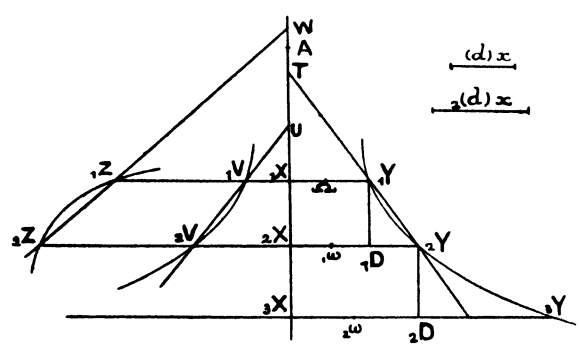
hence, when the difference vanishes, we have

$$3xx : aa = {}_1X\,{}_1Y : T\,{}_1X.$$

But if it is desired to retain dy and dx in the calculation, so that they may represent non-evanescent quantities even in the ultimate case, let any assignable straight line be taken as (dx), and let the straight line which bears to (dx) the ratio of y or ${}_1X\,{}_1Y$ to ${}_1XT$ be called (dy); in this way dy and dx will always be assignables bearing to one another the ratio of $D\,{}_2Y$ to $D\,{}_1Y$, which latter vanish in the ultimate case.

[Leibniz here gives a correction for a passage in the *Acta Eruditorum,* which is unintelligible without the context.]

On these suppositions, all the rules of our algorithm, as set out in the *Acta Eruditorum* for October 1684, can be proved without much trouble.



Let the curves YY, VV, ZZ be referred to the same axis AXX; and to the abscissae $A\,{}_1X\ (=x)$ and $A\,{}_2X\ (=x+dx)$ let there correspond the ordinates ${}_1X\,{}_1Y\ (=y)$ and ${}_2X\,{}_2Y\ (=y+dy)$, and also the ordinates ${}_1X\,{}_1V\ (=v)$ and ${}_2X\,{}_2V\ (=v+dv)$, and the ordinates

${}_1X\,{}_1Z$ ($=z$) and ${}_2X\,{}_2Z$ ($=z+dz$). Let the chords ${}_1Y\,{}_2Y$, ${}_1V\,{}_2V$, ${}_1Z\,{}_2Z$, when produced meet the axis AXX in T, U, W. Take any straight line you will as $(d)x$, and, while the point ${}_1X$ remains fixed and the point ${}_2X$ approaches ${}_1X$ in any manner, let this remain constant, and let $(d)y$ be another line which bears to $(d)x$ the ratio of y to ${}_1XT$, or of dy to dx; and similarly, let $(d)v$ be to $(d)x$ as v to ${}_1XU$ or dv to dx; also let $(d)z$ be to $(d)x$ as z to ${}_1XW$ or dz to dx; then $(d)x$, $(d)y$, $(d)z$, $(d)w$ will always be ordinary or assignable straight lines.

Nor for *Addition and Subtraction* we have the following:

If $y-z=v$, then $(d)y-(d)z=(d)v$.

This I prove thus: $y+dy-z-dz=v+dv$, (if we suppose that as y increases, z and v also increase; otherwise for decreasing quantities, for z say, $-dz$ should be taken instead of dz, as I mentioned once before); hence, rejecting the equals, namely $y-z$ from one side, and v from the other, we have $dy-dz=dv$, and therefore also $dy-dz:dx=dv:dx$. But $dy:dx$, $dz:dx$, $dv:dx$ are respectively equal to $(d)y:(d)x$, $(d)z:(d)x$, and $(d)v:(d)x$. Similarly, $(d)z:(d)y$ and $(d)v:(d)y$ are respectively equal to $dz:dy$ and $dv:dy$. Hence, $(d)y-(d)z, :(d)x=(d)v:(d)x$; and thus $(d)y-(d)z$ is equal to $(d)v$, which was to be proved; or we may write the result as $(d)v:(d)y=1-(d)z:(d)y$.

This rule for addition and subtraction also comes out by the use of our postulate of a common calculation, when ${}_1X$ coincides with ${}_2X$, and ${}_1YT$, ${}_1YU$, ${}_1YW$ are the tangents to the curves YY, VV, ZZ. Moreover, although we may be content with the assignable quantities $(d)y$, $(d)v$, $(d)z$, $(d)x$, etc., since in this way we may perceive the whole fruit of our calculus, namely a construction by means of assignable quantities, yet it is plain from what I have said that, at least in our minds, the unassignables dx and dy may be substituted for them by a method of supposition even in the case when they are evanescent; for the ratio $dy:dx$ can always be reduced to the ratio $(d)y:(d)x$, a ratio between quantities that are assignable or undoubtedly real. Thus we have in the case of tangents $dv:dy=1-dz:dx$, or $dv=dy-dz$.

Multiplication. Let $ay=xv$, then $a(d)y=x(d)v+v(d)x$.

Proof. $ay+a\,dy=x+dx,\ v+dv=xv+x\,dv+v\,dx+dx\,dv$;

and, rejecting the equals ay and xy from the two sides,

$$a\,dy = x dv + v\,dx + dx\;dv,$$

or

$$\frac{a\;dy}{dx} = \frac{x\,dv}{dx} + v + dv\;;$$

and transferring the matter, as we may, to straight lines that never become evanescent, we have

$$\frac{a(d)y}{(d)x} + \frac{x(d)y}{(d)x} + v + dv\;;$$

so that, since it alone can become evanescent, dv is superfluous, and in the case of the vanishing differences, as in that case $dv = 0$, we have

$$a(d)y = x(d)v + v(d)x, \text{ as was stated,}$$

or $$(d)y : (d)x = x + v, : a.$$

Also, since $(d)y : (d)x$ always $= dy : dx$, it will be allowable to suppose this is true in the case when dy, dx become evanescent, and to say that $dy : dx = x + v : a$, or $a\,dy = x\,dv + v\,dx$.

Division. Let $z : a = v : x$, then $(d)z : a = v(d)x - x(d)y, : xx$.

Proof $$z + dz : a = v + dv, : ,x + dx\;;$$

or clearing of fractions, $xz + xdz + zdx + dzdx = av + adv$; taking away the equals xz and av from the two sides, and dividing what is left by dx, we have

$$a\,dv - x\,dz, : dx = z + dz,$$

$$\text{or } a(d)v - x(d)z, : dx = z + dz\;;$$

and thus, only dz, which can become evanescent, is superfluous. Also, in the case of vanishing differences, when ${}_1X$ coincides with ${}_2X$, since in that case $dz = 0$, we have

$$a(d)v - x(d)z, : (d)x = z = av : x\;;$$

whence, (as was stated) $(d)z = ax(d)v - av(d)x, : xx$,

or $$(d)z : (d)x = (a : x)\,(d)v : (d)x - av : xx.$$

Also, since $(d)z : (d)x$ is always equal to $dz : dx$, on all other occasions, it is allowable to suppose this to be so also when dz, dv, dx are evanescent, and to put

$$dz : dx = ax\;dv - av\;dx, : xx$$

For *Powers*, let the equation be $a^{\underline{n-e}}x^e = y^n$, then

$$\frac{(d)\,y}{(d)\,x} = \frac{e.x^{\underline{e-1}}}{n.y^{\underline{n-1}}}\;;$$

and this I will prove in a manner a little more detailed than those above, thus:

$$a^{n-e},\frac{1}{1}x^{e}+\frac{e}{1}x^{e-1}dx+\frac{e,e-1}{1,2}x^{e-2}dxdx+\frac{e,e-1,e-2}{1,2,3}x^{e-3}dxdxdx$$

(and so on until the factor $e-e$ or 0 is reached)

$$=\frac{1}{1}y^{n}+\frac{n}{1}y^{n-1}dy+\frac{n,n-1}{1,2}y^{n-2}dydy+\frac{n,n-1,n-2}{1,2,3}y^{n-3}dydydy$$

(and so on until the factor $n-n$ or 0 is reached);
take away from the one side $a^{n-e}x^{e}$, and from the other side y^{n}, these being equal to one another, and divide what is left by dx, and lastly, instead of the ratio $dy:dx$, between the two quantities that continually diminish, substitute the ratio that is equal to it, $(d)y:(d)x$, a ratio between two quantities, of which one, $(d)x$, always remains the same during the time that the differences are diminishing, or while ${}_2X$ is approaching the fixed point ${}_1X$ and we have

$$\frac{e}{1}x^{e-1}+\frac{e,e-1}{1,2}x^{e-2}dx+\frac{e,e-1,e-2}{1,2,3}x^{e-3}dxdx+\text{etc.}$$

$$=\frac{n}{1}y^{n-1}\frac{(d)y}{(d)x}+\frac{n,n-1}{1,2}y^{n-2}\frac{(d)y}{(d)x}dy+\frac{n,n-1,n-2}{1,2,3}y^{n-3}\frac{(d)y}{(d)x}dydy+\text{etc.}$$

Now, since by the postulate there is included in this general rule the case also in which the differences become equal to zero, that is when the points ${}_2X$, ${}_2Y$ coincide with the points ${}_1X$, ${}_1Y$ respectively; therefore, in that case, putting dx and dy equal to 0, we have

$$\frac{e}{1}x^{e-1}=\frac{n}{1}y^{n-1}\frac{(d)y}{(d)x},$$

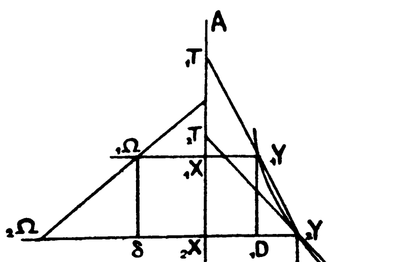
the remaining terms vanishing, or $(d)y:(d)x=e.x^{x-1}:n\ y^{n-1}$. Moreover, as we have explained, the ratio $(d)y:(d)x$ is the same as the ratio of y, or the ordinate ${}_1X{}_1Y$, to the subtangent ${}_1XT$, where it is supposed that T_1Y touches the curve in ${}_1Y$.

This proof holds good whether the powers are integral powers or roots of which the exponents are fractions. Though we may also get rid of fractional exponents by raising each side of the equation to some power, so that e and n will then signify nothing else but powers with rational exponents, and there will be no need of a series proceeding to infinity. Moreover, at any rate, it will be permissible, by means of the explanation given above, to return to the unassignable quantities dy and dx, by making in the case of evanescent differences, as in all other cases, the supposition that the ratio of the evanescent quantities dy and dx is equal to the ratio

of $(d)y$ and $(d)x$, because this supposition can always be reduced to an undoubtable truth.

Thus far the algorithm has been demonstrated for differences of the first order: now I will proceed to show that the same method will hold good for the differences of the differences. For this purpose, take three ordinates, ${}_1X{}_1Y$, ${}_2X{}_2Y$, ${}_3X{}_3Y$, of which ${}_1X{}_1Y$ remains constant, but ${}_2X{}_2Y$ and ${}_3X{}_3Y$ continually approach ${}_1X{}_1Y$ until finally they both coincide with it simultaneously; which will happen if the speed with which ${}_3X$ approaches ${}_1X$ is to the speed with which ${}_2X$ approaches ${}_1X$ is in the ratio of ${}_1X{}_3X$ to ${}_1X{}_2X$. Also let two straight lines be assigned, $(d)x$ always constant for any position of ${}_2X$, and ${}_2(d)x$ for any position of ${}_3X$; also let $(d)y$ always be to $(d)x$ as $D{}_2Y$ is to ${}_1X{}_2X$, or as y (i. e., ${}_1X{}_1Y$) is to ${}_1XT$; thus, while $(d)x$ remains always the same, $(d)y$ will be altered as ${}_2X$ approaches ${}_1X$; similarly, let ${}_2(d)y$ be to ${}_2(d)x$ as ${}_2D{}_3Y$ to ${}_2X{}_3X$ or as $y+dy$ (i. e., ${}_2X{}_2Y$) to ${}_2X{}_2T$; thus while ${}_2(d)x$ remains constant, ${}_2(d)y$ will be altered as ${}_3X$ approaches ${}_1X$.

Also let $(d)y$ be always taken in the varying line ${}_2X{}_2Y$, and let ${}_2X{}_1\omega$ be equal to $(d)y$, and similarly take ${}_2(d)y$ in the line ${}_3X{}_3Y$, and let ${}_3X{}_2\omega$ be equal to ${}_2(d)y$. Thus, while ${}_2X$ and ${}_3X$ continually approach to the straight line ${}_1X{}_1Y$, ${}_2X{}_1\omega$ and ${}_3X{}_2\omega$ continually approach it also, and finally coincide with it at the same time as



${}_2X$ and ${}_3X$. Further, let the point in the ordinate ${}_1X{}_1Y$, which ${}_1\omega$ continually approaches and with which it at last coincides, be marked, and let it be Ω; then ${}_1X\Omega$ is the ultimate $(d)y$, which bears to $(d)x$ the ratio of the ordinate ${}_1X{}_1Y$ to the subtangent ${}_1XT$, where it is supposed that $T{}_1X$ touches the curve in ${}_1Y$, because then indeed ${}_1Y$ and ${}_2Y$ coincide. Now, since all this can be done,

no matter where $_1$Y may be taken on the curve, it is evident that a curve $\Omega\Omega$ will be produced in this way, which is the differentrix of the curve YY; just as, conversely, the curve YY is the summatrix curve of $\Omega\Omega$, as can be readily demonstrated.

By this method, the calculus may be demonstrated also for the differences of the differences.

Let $_1X\,_1Y$, $_2X\,_2Y$, $_3X\,_3Y$ be three ordinates, of which the values are y, $y+dy$, $y+dy+ddy$, and let $_1X\,_2X$ (dx) and $_2X\,_3X$ $(dx+ddx)$ be any distances, and $D\,_2Y$ (dy) and $_2D\,_3Y$ $(dy+ddy)$ the differences. Now the difference between $(d)y$ and $_2(d)y$, or between $_1X\Omega$ and $_2X\,_2\Omega$ is $\delta\,_2\Omega$, and that between $_1X\,_2X$ and $_2X\,_3X$ is ddx; also let

$(d)dx : (d)x = dx : {}_2(d)x,$ [74] and similarly let

$(d)dy : (d)y = {}_2\Omega\delta : {}_1X\,_2X$ or ${}_1X\Omega : {}_1XT.$

Now, for the sake of example, let us take $ay=xv$. Then we have $a\,dy = x\,dv + v\,dx + dx\,dv$, as has been shown above; and similarly,

$$\begin{aligned} a\,dy + a\,ddy = {} & (x+dx)(dv+ddv) + (v+dv)(dx+ddx) \quad {}^{75} \\ & + (dx+ddx)(dv+ddv) \\ = {} & x\,dv + x\,ddv + dx\,dv + dx\,ddv + v\,dx + v\,ddx \\ & + dv\,dx + dv\,ddx + dx\,dv + dx\,ddv \\ & + ddx\,dv + ddx\,ddv. \end{aligned}$$

Taking away $a\,dy$ from one side, and $x\,dx + v\,dx + dx\,dv$ from the other, there will be left in any case

$$\frac{ddy}{ddx} = \frac{x\,ddy}{a\,ddx} + \frac{v}{a} + \frac{2\,dx\,dv}{a\,ddx} + \frac{2\,dv}{a} + \frac{2\,dx\,ddx}{a\,ddx} + \frac{ddv}{a}$$

In this it is evident that the ratio between ddy and ddx can be expressed by the ratio of the straight line $(d)dy$ to $(d)x$, the straight line assumed above, which we have supposed to remain constant as $_2X$ and $_3X$ approach $_1X$. Also, since $(d)dx$, (since it bears an assignable ratio to $(d)x$, however nearly $_2X$ approaches to $_1X$, or

[74] This makes $(d)dx$ an inassignable. It may be a misprint due to a slip of Leibniz, or of Gerhardt in transcription; for there is no similarity between it and the statement in the next line. I cannot however offer any feasible suggestion for correction.

[75] This is quite wrong. Leibniz has evidently substituted $x+dx$ for x, etc.; which is not legitimate unless $_3X_3Y$ is taken as $y+dy+d(y+dy)$, and so on; even then fresh difficulties would be introduced. As it stands, this line should read

$$a\,dy + a\,ddy = x(dv+ddv) + v(dx+ddx) + (dx+ddx)(dv+ddv).$$

On account of this error and that noted above, there is not much profit in considering the remainder of this passage.

however much dx, the difference between the abscissae, is diminished), is not evanescent, even when, finally, dx and ddx, dv and ddv, are all supposed to be zero. In the same way, the ratio of ddv to ddx may be expressed by the ratio of an assignable straight line $(d)dv$ to the assumed constant $(d)x$; and even the ratio of $dv\,dx$ to $a\,ddx$ may be so expressed; for, since $dv : dx = (d)v : (d)x$, therefore $dv\,dx : dx\,dx = (d)v : (d)x$. Hence, if a new straight line, $(dd)x$, is assumed to be such that $a\,ddx : dx\,dx = (dd)x s (d)x$, then the new straight line will be assignable, even though dx, ddx, etc. become evanescent. Since therefore $dv\,dx : dx\,dx = (d)v : (d)x$ and $dv\,dx : a\,ddx = (d)x : (dd)x$, it follows that $dv\,dx : a\,ddx = (d)v : (dd)x$, an thus at length there is prouced an equation that is freed as far as possible from those ratios that might become evenescent, namely,

$$\frac{(d)dy}{(d)dx} = \frac{x}{a}\frac{(d)dy}{(d)dx} + \frac{y}{a} + \frac{2\,(d)y}{(dd)x} + \frac{2\,dv}{a} + \frac{2dx}{a}\frac{(d)dy}{(d)dx} + \frac{ddv}{a}$$

Thus far all the straight lines have been considered to be assignable so long as ${}_1X$ and ${}_2X$ do not coincide; but in the case of coincidence, dv and ddv are zero, and we have

$$\frac{(d)dy}{(d)dx} = \frac{x}{a}\frac{(d)dv}{(d)dx} + \frac{v}{a} + \frac{2\,(d)y}{(dd)x} + \frac{0}{a} + \frac{2\,(d)dv}{(d)dx}\frac{0}{a} + \frac{0}{a},$$

or, omitting terms equal to zero,

$$\frac{(d)dy}{(d)dx} = \frac{x}{a}\frac{(d)dv}{(d)dx} + \frac{v}{a} + \frac{2\,(d)y}{(dd)x}.$$

Hence, if dx, ddx, dv, ddv, dy, ddy, are by a certain fiction imagined to remain, even when they become evanescent, as if they were infinitely small quantities (and in this there is no danger, since the whole matter can be always referred back to assignable quantities), then we have in the case of coincidence of the point ${}_1X$ and ${}_2X$ the equation

$$\frac{ddv}{ddx} = \frac{x}{a}\frac{ddy}{ddx} + \frac{v}{a} + \frac{2}{a}\frac{dx\,dy}{ddx}.$$

J. M. Child.

Derby, England.